A1

i) $d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$

$x = y \iff d(x,y) = 0$

$d(x,y) = d(y,x)$

z.Z.: $d(x,y) \leq d(x,z) + d(z,y)$

Fall $z \neq x$ oder $z \neq y$

$d(x,y) \leq 1 \leq d(x,z) + d(z,y)$

Fall $z = x = y$

$d(y,y) = 0$

ii) $X = \{0,1\}^n$, $x,y \in X$, $d(x,y) = \#\{\ell \in \{1,\dots,n\}, x_\ell \neq y_\ell\}$

$d(x,y) = 0 \iff x = y$

$d(x,y) = d(y,x)$ ✓

$$d(x,y) = \sum_{i \in [n]} \mathbb{1}_{x_i \neq y_i} \leq \sum_{i \in [n]} \mathbb{1}_{x_i \neq z_i} + \sum_{i \in [n]} \mathbb{1}_{z_i \neq y_i}$$

Für jedes $i \in [n]$ und $z \in X$

ist $\mathbb{1}_{x_i \neq y_i} \leq \mathbb{1}_{x_i \neq z_i} + \mathbb{1}_{z_i \neq y_i}$ (siehe (i)

$\Rightarrow d(x,y) \leq d(x,z) + d(z,y)$

$f_2\left(\lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = f_2\begin{pmatrix} \lambda x_1 \\ \lambda x_2 \end{pmatrix} = |\lambda x_1| + |\lambda x_2| = |\lambda|(|x_1| + |x_2|) = |\lambda| f_2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

A2

$f_h : \mathbb{R}^2 \to \mathbb{R}$

i)

$f_1(X) = \max\{x_1, x_2\}$

$f_1\begin{pmatrix}-1\\-2\end{pmatrix} = -1 < 0$

ii)

$f_2(X) = |x_1| + |x_2|$

Seien $\begin{pmatrix}x_1\\x_2\end{pmatrix}, \begin{pmatrix}x_3\\x_4\end{pmatrix} \in \mathbb{R}^2$, $\lambda \in \mathbb{R}$.

(1)

$f_2\begin{pmatrix}x_1\\x_2\end{pmatrix} = |x_1| + |x_2| \geq 0$

(2)

$f_2\begin{pmatrix}x_1\\x_2\end{pmatrix} = 0 \iff \underbrace{|x_1|}_{\geq 0} + \underbrace{|x_2|}_{\geq 0} = 0 \iff |x_1| = |x_2| = 0$

$\iff x_1 = x_2 = 0$

iii)

$f_3(X) = \left(\sqrt{|x_1|} + \sqrt{|x_2|}\right)^2$

iv)

$f_4(X) = 1 + |x_1|^2 + |x_2|^2$

(2)

$f_2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Leftrightarrow \underbrace{|x_1|}_{\geq 0} + \underbrace{|x_2|}_{\geq 0} = 0 \Leftrightarrow |x_1| = |x_2| = 0$

$\Leftrightarrow x_1 = x_2 = 0$

iii

$f_3(x) = \left(\sqrt{|x_1|} + \sqrt{|x_2|}\right)^2$

$f_3\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = f_3\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \left(\sqrt{|1|} + \sqrt{|1|}\right)^2$

$= 2^2 = 4$

$f_3\begin{pmatrix} 1 \\ 0 \end{pmatrix} + f_3\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \left(\sqrt{|1|} + \sqrt{|0|}\right)^2$

$+ \left(\sqrt{|0|} + \sqrt{|1|}\right)^2 = 1^2 + 1^2 = 2$

$f_3\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) > f_3\begin{pmatrix} 1 \\ 0 \end{pmatrix} + f_3\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

iv

$f_4(x) = 1 + |x_1|^2 + |x_2|^2$

$f_4\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 + |1|^2 + |1|^2 = 3$

$f_4\begin{pmatrix} 2 \\ 2 \end{pmatrix} = 1 + |2|^2 + |2^2| = 1 + 4 + 4 = 9$

$f_4\begin{pmatrix} 2 \\ 2 \end{pmatrix} \neq 2 f_4\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$x = y \iff d(x,y) = 0$

$d(x,y) = d(y,x)$

z.Z.: $d(x,y) \leq d(x,z) + d(z,y)$

Fall: $z \neq x$ oder $z \neq y$

$d(x,y) \leq 1 \leq d(x,z) + d(z,y)$

Fall $z = x = y$

$d(y,y) = 0$

$d(x,y) = \sum_{i \in [n]} 1\!\!1_{x_i \neq y_i}$

Für jedes $i \in [n]$ und $z \in X$

ist $1\!\!1_{x_i \neq y_i} \leq 1\!\!1_{x_i \neq z_i} + 1\!\!1_{z_i \neq y_i}$ (siehe (i))

$\Rightarrow d(x,y) \leq d(x,z) + d(z,y)$

(3)

$$f_2\left(\lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = f_2\begin{pmatrix} \lambda x_1 \\ \lambda x_2 \end{pmatrix} = |\lambda x_1| + |\lambda x_2| = |\lambda|(|x_1| + |x_2|) = |\lambda| f_2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(4)

$$f_2\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}\right) = f_2\begin{pmatrix} x_1 + x_3 \\ x_2 + x_4 \end{pmatrix} = |x_1 + x_3| + |x_2 + x_4| \leq |x_1| + |x_2| + |x_3| + |x_4| = f_2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + f_2\begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$$