Üb 2

A1

1) $a > 0$. $f(x) = x^2$

1)

$x_n = a + \frac{k(b-a)}{n}$ $\quad n = \{0, n\}$

$$S(P_n, f) = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_{k-1}) \Delta x = \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\lim_{n \to \infty} \sum_{k=1}^{n} \left(a + \frac{(k-1)(b-a)}{n} \right)^2 \cdot \frac{b-a}{n} =$$

$$\lim_{n \to \infty} \sum_{k=1}^{n} \left(a^2 + \frac{2(k-1)a(b-a)}{n} + \frac{(k-1)^2(b-a)^2}{n^2} \right) \frac{(b-a)}{n} =$$

$$\lim_{n \to \infty} \left(n \cdot a^2 + \frac{2a(b-a)}{n} \left(\frac{(n-1) \cdot n}{2} \right) + \frac{(b-a)^2}{n^2} \frac{(n-1) \cdot n(2n-3)}{6} \right) \frac{(b-a)}{n} =$$

$$\lim_{n \to \infty} \left(a^2(b-a) + \frac{\cancel{2} a(b-a)^2 (n-1)}{\cancel{2} n} + \frac{(b-a)^3}{} \frac{(n-1)(2n-3)}{6} \right) =$$

$$= a^2(b-a) + a(b-a)^2 + \frac{(b-a)^3}{3} \quad n^2 \quad = \frac{(b-a)}{3} (\cancel{3a^2} + 3ab - \cancel{3a^2} + b^2 - 2ab + a^2)$$

Üb 2

A1

ii)

$$= \frac{(b+a)}{3}\left(a^2 - ab + b^2\right) = \frac{b^3 - a^3}{3}$$

3) $$\int_a^b x^2\,dx = \left[\frac{x^3}{3}\right]_a^b = \frac{b^3}{3} - \frac{a^3}{3}$$

iii)

Üb 2

A1

i/ $a > 0$. $f(x) = x^2$

1)

$$X_h = a + \frac{k(b-a)}{h} \qquad n = \{0, n\}$$

$$S(P_n, f) = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_{k-1})\,\Delta x = \qquad \sum_{k=1}^{n} k^2 = \frac{n(k+1)(2k-1)}{6}$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(a + \frac{(k-1)(b-a)}{h}\right)^2 \cdot \frac{b-a}{n} =$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(a^2 + \frac{2(k-1)(b-a)}{h} + \frac{(k-1)^2(b-a)^2}{h^2}\right) \frac{(b-a)}{h} =$$

$$\lim_{n\to\infty} \left(h \cdot a^2 + \frac{2a(b-a)}{h}\left(\frac{(h-1)\cdot h}{2}\right) + \frac{(b-a)^2}{h^2} \frac{(h-1)\cdot h(2h-3)}{6}\right)\frac{(b-a)}{h}$$

$$\lim_{n\to\infty} \left(a^2(b-a) + \frac{\cancel{2}a(b-a)^2(h-1)}{\cancel{2}h} + \frac{(b-a)^3}{h^2} \frac{(n-1)(2n-3)}{6}\right) =$$

$$= a^2(b-a) + a(b-a)^2 + \frac{(b-a)^3}{3} = \frac{(b-a)}{3}\left(\cancel{3a^2} + 3ab - \cancel{3a^2} + b^2 - 2ab + a^2\right) =$$

A2

i) z.z. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, $c \in [a,b]$

P_n, $x_m = c$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) = \lim_{n\to\infty}\left(\left(\sum_{k=1}^{m} f(\xi_k)(x_k - x_{k-1})\right) + \sum_{k=m+1}^{n} f(\xi_k)(x_k - x_{k-1})\right)$$

$$= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

ii) z.z. $\int_a^b (\alpha f + \beta g)(x)\,dx = \alpha\int_a^b f(x)\,dx + \beta\int_a^b g(x)\,dx$

$$\int_a^b (\alpha f + \beta g)(x)\,dx = \int_a^b (\alpha f(x) + \beta g(x))\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} (\alpha f(\xi_k) + \beta g(\xi_k))(x_k - x_{k-1})$$

$$= \lim_{n\to\infty}\left(\alpha\sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) + \beta\sum_{k=1}^{n} g(\xi_k)(x_k - x_{k-1})\right) = \alpha\int_a^b f(x)\,dx + \beta\int_a^b g(x)\,dx$$

$1.1)\ \int_1^e \frac{\ln(x)^3}{x}\,dx = \int_0^1 t^3\,dt = \left[\frac{1}{4}t^4\right]_0^1 = \frac{1}{4}$

$t = \ln(x) \quad \in [0,1]$

$dt = \frac{dx}{x}$

$1.2)\ \int_5^6 \frac{2}{x^2-3x-4}\,dx = 2\int_5^6 \frac{1}{\underbrace{x^2-3x-4}_{(x+1)(x-4)}}\,dx = 2\int_5^6 \frac{1}{(x+1)(x-4)}\,dx = \int_5^6 2\,\frac{1}{(x+1)}\cdot\frac{1}{(x-4)}\,dx = 2\int_5^6 \frac{\frac{1}{5}}{x+1} - \frac{\frac{1}{5}}{x-4}\,dx =$

$= -\frac{2}{5}\int_5^6 \frac{1}{x+1}\,dx + \frac{2}{5}\int_5^6 \frac{1}{x-4}\,dx =$

$= -\frac{2}{5}\left[\ln(x+1)\right]_5^6 + \frac{2}{5}\left[\ln(x-4)\right]_5^6$

$= -\frac{2}{5}\left(\ln(7) - \ln(6)\right) + \frac{2}{5}\left(\ln(2) - \ln(1)\right)$

$= \frac{2}{5}\ \ln\left(\frac{6\cdot 2}{7\cdot 1}\right) = \frac{2}{5}\cdot\ln\left(\frac{12}{7}\right)$

$\frac{1}{(x+1)\cdot(x-4)} = \frac{\quad}{x+1} \pm \frac{\quad}{x-4}$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{k^2+n^2}$$

$$f: [0,1] \to \mathbb{R}, \quad f(x) = \frac{1}{x^2+1}$$

$$P_n: \; x_0 = 0 < \dots < x_n = 1 \qquad x_k = \frac{k}{n}$$

$$|P_n| = x_{k+1} - x_k = \frac{k+1}{n} - \frac{k}{n} = \frac{1}{n} \xrightarrow{n\to\infty} 0$$

$$\xi_k = x_k$$

$$\sum_{k=1}^{n} f(\xi_k)\cdot(x_k - x_{k-1})$$

$$= \sum_{k=1}^{n} \frac{1}{\frac{k^2}{n^2}+1} \, \frac{1}{n} = \sum_{k=1}^{n} \frac{n^2}{k^2+n^2} \, \frac{1}{n} = \sum_{k=1}^{n} \frac{n}{k^2+n^2}$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{k^2+n^2} = \int_0^1 f(x)\, dx = \int_0^1 \frac{1}{x^2+1}\, dx = \left[\arctan(x)\right]_0^1$$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

f(ii)

$$\lim_{n\to\infty}\sum_{k=n+1}^{2n}\frac{n}{k^2} = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{(n+k)^2} = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\cdot\frac{n^2}{(n+k)^2} = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\frac{1}{\left(1+\frac{k}{n}\right)^2}$$

$$= \int_0^1 \frac{1}{(1+x)^2}\,dx = \left[-\frac{1}{1+x}\right]_{x=0}^{x=1} = -\frac{1}{2}-(-1) = \frac{1}{2} \,//$$

$$\prod_{n=1}^{\infty} \frac{4n^2}{4n^2-1} = \frac{\pi}{2}$$

$$I_{2n} = \left(\prod_{k=1}^{n} \frac{2k-1}{2k}\right) \frac{\pi}{2}$$

$$I_{2n+1} = \left(\prod_{k=1}^{n} \frac{2k}{2k+1}\right) \cdot 1$$

$$\frac{I_{2n+1}}{I_{2n}} = \left(\prod_{k=1}^{n} \frac{2k}{2k+1} \frac{2k}{2k-1}\right) \frac{2}{\pi} = \frac{2}{\pi} \prod_{k=1}^{n} \frac{4k^2}{4k^2-1}$$

$$\sin^{2n+1}(x) \leq \sin^{2n}(x) \leq \sin^{2n-1}(x)$$

$$I_{2n+1} \leq I_{2n} \leq I_{2n-1} \qquad I =$$

$$\frac{2n}{2n+1} \frac{\cancel{I_{2n+1}}}{\cancel{I_{2n+1}}} = \frac{I_{2n+1}}{I_{2n-1}} \leq \frac{I_{2n+1}}{I_{2n}} \leq 1$$

$$\lim_{n \to \infty} \sum_{k=1}^{n} \frac{n}{k^2+n^2}$$

$P \qquad x_0 = 0 < \quad < x_n = 1 \qquad x_k = \frac{k}{n}$

$$I_m = \int_0^{\pi/2} \sin(x)^m \, dx = \Big[\sin^{m-1}(x) \, (-\cos(x)) \Big]_0^{\pi/2} + \int_0^{\pi/2} (m-1) \sin^{m-2}(x) \cos^2(x) \, dx$$

$$= (m-1) \int_0^{\pi/2} \sin(x)^{m-2} \, (1 - \sin^2(x)) \, dx$$

$$= (m-1) \int_0^{\pi/2} \sin(x)^{m-2} \, dx - (m-1) \int_0^{\pi/2} \sin(x)^m \, dx$$

$$I_m = (m-1) \, I_{m-2} - (m-1) \, I_m$$

$$I_m = \frac{m-1}{m} \, I_{m-2}$$

$$I_0 = \frac{\pi}{2}$$

$$I_1 = 1$$