Aufgabe 2

$a_n x^n + a_{n-1} x^{n} + \dots + a_1 x + a_0$

Fundamental

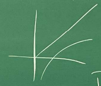
a) $\lim\limits_{n\to\infty} n^s \cdot e^{-n} = \lim\limits_{n\to\infty} e^{\ln(n^s)} \cdot e^{-n} = \lim\limits_{n\to\infty} e^{s\cdot\ln(n)-n} = 0$

$b_n := s\cdot\ln(n)-n$

$b_{n+1} = s\cdot\ln(n+1)-n-1$

$b_{n+1}-b_n = s\cdot\ln(n+1)-\cancel{n}-1-s\cdot\ln(n)+\cancel{n}$

$= s(\ln(n+1)-\ln(n))-1$



$\boxed{\ln(n) < n}$

$n < e^n$

$|b_{n+1}-b_n|$ $= s\left(\ln\left(\frac{n+1}{n}\right)\right)-1 \not\longrightarrow 0$ für $n\to\infty,\ n\neq 0 \Rightarrow$ keine Nullfolge

Divergenz

b) $\lim\limits_{n\to\infty} \frac{\ln(n+a)}{\ln(n)} = \frac{\ln\left(n\cdot\left(1+\frac{a}{n}\right)\right)}{\ln n} = \frac{\ln(n)+\ln\left(1+\frac{a}{n}\right)}{\ln(n)}$

$\Leftrightarrow \lim\limits_{n\to\infty} 1+\frac{\ln\left(1+\frac{a}{n}\right)}{\ln(n)} = 1+\lim\limits_{n\to\infty}\frac{\ln\left(1+\frac{a}{n}\right)}{\ln(n)} = 1+\frac{\ln\left(\lim\limits_{n\to\infty} 1+\frac{a}{n}\right)}{\lim\limits_{n\to\infty}\ln(n)}$

ln ist stetig

$= 1+\frac{\ln(1)}{\lim\limits_{n\to\infty}\ln(n)} = 1$

$\forall z\in\mathbb{C}: \quad \exp(z) = \lim\limits_{n\to\infty}\left(1+\frac{z}{n}\right)^n$

$a_n = \sum_{k=0}^{n}\frac{z^k}{k!} - \left(1+\frac{z}{n}\right)^n = \dots = \sum_{k=0}^{n}\underbrace{\left(1-\frac{n!}{(n-k)!\cdot n^k}\right)}\cdot\frac{z^k}{k!}$

$1-\prod_{m=1}^{k}\frac{n-m+1}{n} = 1-\prod_{m=0}^{k}\underbrace{\left(1-\frac{m}{n}\right)}_{<1} \leq \frac{k\cdot(k-1)}{2n}$

$\prod_{m=0}^{k}\left(1-\frac{m}{n}\right) \leq \sum_{m=0}^{k}\frac{m}{n} = \frac{k\cdot(k-1)}{2n}$

$|a_n| \leq \sum_{k=0}^{n}\frac{k\cdot(k-1)}{2n}\cdot\frac{|z|^k}{k!} = \sum_{k=2}^{n}\frac{k\cdot(k-1)}{2n}\cdot\frac{|z|^k}{k!} = \sum_{k=2}^{n}\frac{|z|^2}{2n}\cdot\frac{|z|^{k-2}}{(k-2)!}$

$\leq \frac{|z|^2}{2n}\cdot\exp(|z|) \xrightarrow{n\to\infty} 0$

Aufgabe 3

$$\forall z \in \mathbb{R} \qquad \exp(z) = \lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n$$

$$\text{"}\leq\text{"}$$

$$\lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\binom{n}{k}\left(\frac{z}{n}\right)^k\right) = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{n!}{(n-k)!\,k!}\left(\frac{z}{n}\right)^k\right) =$$

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{(n-k+1)\cdot\ldots\cdot n}{n^k}\cdot\frac{z^k}{k!}\right) = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{(n-k+1)}{n}\cdot\ldots\cdot\frac{n}{n}\cdot\frac{z^k}{k!}\right) \underset{n\geq M}{\geq} \lim_{M\to\infty}\lim_{n\to\infty}\left(\sum_{k=1}^{M}\frac{(n-k+1)}{n}\cdot\ldots\cdot\frac{n}{n}\,\frac{z^k}{k!}\right)$$

$$= \sum_{k=1}^{M}\lim_{n\to\infty}\left(\underbrace{\frac{n-k+1}{n}}_{\to 1}\cdot\ldots\cdot\underbrace{\frac{n}{n}}_{\to 1}\,\frac{z^k}{k!}\right) = \underbrace{\sum_{k=1}^{M}\frac{z^k}{k!}}_{S_M \longrightarrow e^z} = \exp(z)$$

$$\text{"}\geq\text{"}$$

$$\left(1+\frac{z}{n}\right)^n = \sum_{k=1}^{n}\binom{n}{k}\frac{z^k}{n^k} = \sum_{k=1}^{n}\underbrace{\frac{(n-k+1)}{n}}_{\leq 1}\cdot\ldots\cdot\underbrace{\frac{n}{n}}_{\leq 1}\,\frac{z^k}{k!} \leq \sum_{k=1}^{n}\frac{z^k}{k!}$$

$$\exp z = \lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n$$

$$\sum^{n}\ldots = \lim_{m}$$

$$\lim_{n\to\infty}\left(\lim_{M\to n}\sum_{k=1}^{M}\frac{z^k}{k!}\right) = e^z$$